Ex5a

Exercise 5

19.11.2021

Reading

Please consider reading the following:

1. David J. Griffiths - introduction to quantum mechanics, second edition, section 4.3.
2. J.J Sakurai, modern quantum mechanics, second eddition, Chapter 3, section 8.

6.1 Clebsch-Gordan Coefficients for $1 \otimes \frac{1}{2}$.

Consider a spin 1/2 particle in a state with orbital angular momentum $\ell = 1$. Construct states of definite total angular momentum from simultaneous eigenstates of orbital angular momentum and spin. Label the eigenstates in the "uncoupled basis" (ie eigenstates of L^2, S^2, L_z, and S_z) by $|\ell s m_\ell m_s\rangle$. Label the states in the "coupled basis" (ie eigenstates of J^2 and J_z) by $|j, m\rangle$.

a) Find the state with maximum j and $m\,(= j_{\max})$ in terms of the $|\ell s m_\ell m_s\rangle$ states.

b) Use $J_- = L_- + S_-$to generate all the $|j_{\max}, m\rangle$ states.

c) Use orthonormality to find the state $|j_{\max} - 1, j_{\max} - 1\rangle$.

d) Use $J_-$to generate all the states $|j_{\max} - 1, m\rangle$.

e) Repeat steps (c) and (d) for smaller j 's as many times as necessary.

f) Check you results with (Table 4.8 of the book Griffiths, introduction to quantum mechanics).

g) What is the expectation value of L_z in the state with $j = 1/2, m = 1/2$? What is the expectation value of S_z in this state?

h) Suppose that this particle moves in an external magnetic field in the z-direction, $\vec{B} = B\vec{e}_z$. Assume the particle is an electron, and take $g = 2$. The Hamiltonian describing the interaction of the electron with the field is

$$H_B = \frac{\mu_B}{\hbar}\vec{B} \cdot (\mathbf{L} + 2\mathbf{S})$$

What is $\langle H_B \rangle$ in each of the eigenstates $|j, m\rangle$?

i) For the eigenstate $j = 1/2, m = 1/2$, what are the possible values of the magnetic energy and what are their probabilities?

6.2 More Clebsch-Gordan

For the following exercise you will need the Clebsch-Gordan tables you can find on `http://pdg.lbl.gov/2002/clebrpp.pdf` .

a) Assume two interacting particles with their respective Spins $S^a = 2$ and $S^b = 1$ that couple to the total spin S. Which states $|S, S_z\rangle$ are possible? Write them in terms of $|S^a, S^a_z\rangle$ and $|S^b, S^b_z\rangle$. What is the probability of finding particle a with $S^a_z = 2\hbar$?

b) A hydrogen atom is in the P state. What are the possible fine structure states? Write them in terms of $|L, m_L\rangle$ and $|s, m_s\rangle$.

c) Lithium-6 has a nuclear spin of $I = 1$, it's ground state is $2S$. The coupling of the nuclear spin and the total orbital momentum J gives rise to the hyperfine structure. What are the possible hyperfine structure states? Compare them to the states of exercise b).

6.3 Hyperfine structure in deuterium

Draw the ground-state hyperfine structure of deuterium including the quantum numbers. The deuteron has a nuclear spin of $I = 1$ and a g-factor of $g_I(D) = 0.857$.

a) What is the value for the quantum number of the corresponds to the ground state?

b) How large is the hyperfine constant A for deuterium. Assume that the magnetic field B_J at the nucleus is the same for hydrogen and deuterium.

c) How large is the ground-state HFS splitting?