Ex5a

Exercise 3

When we derived the solution to the hydrogen problem using the Schrödinger equation, we assumed the nucleus (proton) to be a non-moving object with infinite mass. For the hydrogen the proton mass really is negligible. But to be correct we have to use the reduced mass $\mu = \frac{m_\mathrm{p} m_\mathrm{e}}{m_\mathrm{p}+m_\mathrm{e}}$ including masses of proton and electron. This leads to slightly different solutions

$$\Psi_{nlm}(r, \vartheta, \varphi) = R_{nl}(r) Y_{lm}(\vartheta\varphi)$$

where

- $R_{nl}(r) = \sqrt{\left(\frac{2Z}{na_0}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}} e^{-\rho/2} \rho^l L_{n-l-1}^{2l+1}(\rho)$,
- $\rho = \frac{2Zr}{na_0}$ (where $a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$ is the atom's Bohr radius),
- $L_{n-l-1}^{2l+1}(\rho)$ are the generalized Laguerre polynomials,
- $Y_{lm}(\vartheta\varphi)$ are the spherical harmonics.

1 Exotic hydrogen like systems

Exotic atoms are atoms in which the electron or the nucleus is substituted by another negative or positive particle. In this exercise we want to investigate how hydrogen like systems scale with atomic number Z and the masses of the shell particle and the nucleus.

1. First of all, we consider a hydrogen atom in which the electron is substituted by a muon (μ^-). The muon is the heavy 'brother' of the electron, it is approximately 206 times as heavy. Because of its high mass, it only exists for roughly 2 µm, before decaying into one electron and two neutrinos. Muonic hydrogen is produced by shooting a muon beam into hydrogen gas. The muons are decelerated by Coulomb interaction and subsequently kick out the shell electron and replace it. The muon predominantly occupies orbitals with the same energy as the previously bound electron.

   (i) Assume that the hydrogen atoms were in their ground state. Which quantum number n of muonic hydrogen ($p\mu^-$) does this correspond to? Give a general equation for the most probable n as a function of the particle's mass.

   (ii) The so formed muonic atom deexcites to the ground state and emits a photon. What is this photon's energy?

(iii) Assume the proton to be a hard sphere of radius R_p. How much larger is the probability of finding a muon within the proton (for ground state only) compared to finding an electron within the proton in regular hydrogen?

2. A muon is captured by a lead atom and deexcites to the ground state by emitting X-ray photons.

   (i) What is the radius of the first Bohr orbit?

   (ii) How strongly will an 1s electron in lead shield the lead nucleus' charge against the muon? Calculate the probability of finding the 1s electron within the radius of the first Bohr orbit of the muon?

3. Another important system is positronium (e^+e^-). It is a bound state of an electron and its antiparticle: the positron.

   (i) What is the binding energy of ground state positronium?

   (ii) In ground state positronium (angular momentum $l = 0$) the spins of both particles couple trough dipole-dipole interaction because each particle is a magnetic dipole. What total angular momenta (total spins) can form?

2 Coupling of angular momentum and spin

A circular current with angular momentum $\vec{L}$ creates a magnetic field. Electrons that orbit a nucleus form such a current. Additionally, they also have a spin $\vec{S}$ resulting in a magnetic dipole moment. The interaction between the self-induced magnetic field and the magnetic dipole leads to a Hamiltonian

$$\hat{H}_{LS} = \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{2m^2c^2}\frac{1}{r^3}\left(\hat{\vec{L}}\cdot\hat{\vec{S}}\right),$$

the spin-orbit coupling.
The base functions

$$\Psi_{n,l,m,m_s=+1/2} = R_{n,l}Y_{l,m}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\Psi_{n,l,m,m_s=-1/2} = R_{n,l}Y_{l,m}\begin{pmatrix}0\\1\end{pmatrix}$$

are to be transformed into a system with j^2, l^2, s^2, M_j:

$$|J, M_J, l, s\rangle = \sum_{m_s=\pm 1/2} \langle l, m, s, m_s | J, M_J\rangle\, |l, m, s, m_s\rangle$$

$$M_J = m_{j_1} + m_{j_2}$$

The Clebsch-Gordan coefficients $\langle j_1, m_{j_1}, \frac{1}{2}, m_{j_2} | J, M_J\rangle$ for the coupling of an angular momentum (j_1, m_{j_1}) and a spin $(1/2, m_{j_2} = \pm 1/2)$ to the total angular momentum (J, M_J) are given as:

J	$m_{j_2} = +\frac{1}{2}$	$m_{j_2} = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\left(\frac{j_1+M_J+\frac{1}{2}}{2j_1+1}\right)^{1/2}$	$\left(\frac{j_1-M_J+\frac{1}{2}}{2j_1+1}\right)^{1/2}$
$j_1 - \frac{1}{2}$	$-\left(\frac{j_1-M_J+\frac{1}{2}}{2j_1+1}\right)^{1/2}$	$\left(\frac{j_1+M_J+\frac{1}{2}}{2j_1+1}\right)^{1/2}$

(a) Show that you can write the base functions for $M_J = m + 1/2$ as:

$$\begin{aligned}
\bar{\Psi}_{n,l,j=l+1/2,m_s} &= \sqrt{\tfrac{l+m+1}{2l+1}} R_{n,l} Y_{l,m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\tfrac{l-m}{2l+1}} R_{n,l} Y_{l,m+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\bar{\Psi}_{n,l,j=l-1/2,m_s} &= -\sqrt{\tfrac{l-m}{2l+1}} R_{n,l} Y_{l,m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\tfrac{l+m+1}{2l+1}} R_{n,l} Y_{l,m+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

(b) Calculate the product $\vec{L} \cdot \vec{S} = L_x S_x + L_y S_y + L_z S_z$, where

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Use $L_\pm = L_x \pm i L_y$ with $L_\pm Y_{l,m} = \hbar \sqrt{l(l+1) - m(m \pm 1)} Y_{l,m\pm1}$.

(c) Calculate the eigenvalue a of $\vec{L} \cdot \vec{S} \,|J, M_J, l, s\rangle = a \,|J, M_J, l, s\rangle$ for $j = l \pm 1/2$ using the functions of problem (a).

(d) Compare the results to the eigenvalues of $\vec{L} \cdot \vec{S} = \frac{1}{2}(J^2 - L^2 - S^2)$ for $j = l \pm 1/2$.