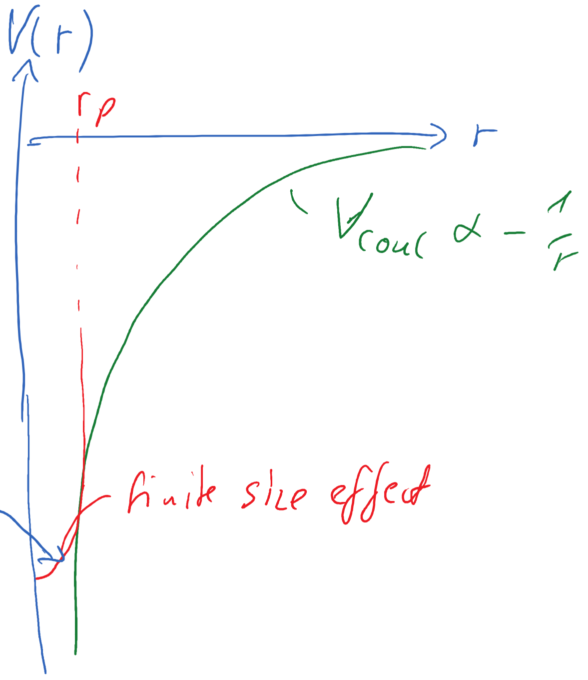
# Exercise 2




Inside the sphere $R_p$

$$V = \frac{e}{8\pi\varepsilon_0 R_p}\left(3 - \frac{r^2}{R_p^2}\right)$$

H atom: $H_0 = \frac{p^2}{2\mu} - \frac{e^2}{4\pi\varepsilon_0 r}$ perfect Coulomb potential

perturbation: inside $r < R_p$

Hamiltonian: $H_{tot} = H_0 + H_1$ ← small, hopefully

$$= \frac{p^2}{2\mu} - e\cdot \frac{e}{8\pi\varepsilon_0 R_p}\left(3 - \frac{r^2}{R_p^2}\right)$$ ← modified Coulomb potential

$$= \underbrace{\frac{p^2}{2\mu} - \frac{e^2}{4\pi\varepsilon_0 r}}_{H_0} + \underbrace{\frac{e^2}{4\pi\varepsilon_0 r} - \frac{e^2}{8\pi\varepsilon_0 R_p}\left(3 - \frac{r^2}{R_p^2}\right)}_{H_1}$$

$H_1$ for $r < R_p$

$H_1 = 0$ for $r \geq R_p$

Ground state w.f. $\psi_{100}(r) = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}$ $a$ = Bohr radius

$\psi_{100}(r) = \frac{1}{\sqrt{\pi} a^3} e \qquad a$ = Bohr radius

1st. order correction for $r < R_p$ : only $H_1$

$$\Delta E = \int_0^{2\pi}\int_0^{\pi}\int_0^{R_p} \psi_1^* H_1 \psi_1 r^2 \underbrace{\sin\Theta\, d\Theta\, d\phi}_{\to 4\pi}$$

$$= 4\pi \int_0^{R_p} \psi_1^* H_1 \psi\, r^2\, dr$$

$$\Rightarrow \Delta E = 4\pi \int_0^{R_p} dr\, r^2 \left(\frac{e^{-r/a}}{\sqrt{\pi a^3}}\right)^2 \left(-3 + \left(\frac{r}{R_p}\right)^2 + \frac{2R_p}{r}\right)\left(\frac{e^2}{8\pi\varepsilon_0 R_p^2}\right)$$

integrate by parts

$$= \dots = \quad R_p \ll a$$

$$\Rightarrow e^{-2r/a} \simeq 1$$

$$\boxed{\Delta E = \frac{e^2}{2\pi\varepsilon_0 R_p a^3} \frac{R_p^5}{5R_p^2}} = \boxed{\frac{e^2}{10\pi\varepsilon_0 a^3} R_p^2}$$

$a_\mu = \frac{a_e}{206}$

$a_H = 5.29177 \cdot 10^{-11}$ m

$a_{\mu p} = 2.56882 \cdot 10^{-13}$ m

$1$ fm $= 10^{-15}$ m

$\Delta E$

| | $R_p = 0.84$ fm | $0.88$ fm |
|---|---|---|
| H | 2.74 neV | 3.01 neV |
| $\mu p$ | 23.975 meV | 26.313 meV $\to \delta\Delta E$ |

2.2. Dirac Equ.

-!-

2.2. Dirac Equ.

$$E_{nj} = m_0 c \left[ 1 + \frac{(Z\alpha)^2}{(n-\delta_j)^2} \right]^{-\frac{1}{2}}$$

$$\delta_j = j + \frac{1}{2} - \sqrt{(j+\tfrac{1}{2})^2 - (Z\alpha)^2}$$

$\hookrightarrow$ binding energy $= E_{nj} - m_0 c^2$ $\in$ D.E. solution

For $S$-$O$-coupling Eq. 2.54, 2.55

$$\left.\begin{array}{l} E_{S\text{-}O} = \frac{\beta}{2} \left\{ j(j+1) - l(l+1) - s(s+1) \right\} \\ \beta = \frac{\hbar^2}{2 m_e^2 c^4} \frac{e^2}{4\pi\varepsilon_0} \frac{1}{(n a_0)^3 \, l (l+\frac{1}{2})(l+1)} \end{array}\right\}$$

$$E_{tot}^{binding} = E_{Bohr} + E_{S\text{-}O} = -c h R_\infty \frac{1}{n^2} + E_{S\text{-}O}$$

| Levels | n | l | s | j | Dirac | Bohr + S-O |
|---|---|---|---|---|---|---|
| $1S_{1/2}$ | 1 | 0 | ½ | ½ | −13.605874 | −13.605693 |
| $2S_{1/2}$ | 2 | 0 | ½ | ½ | −3.401 $\boxed{480}$ | −3.401 233 |
| $2P_{1/2}$ | 2 | 1 | ½ | ½ | $\boxed{480}$ | 456 |
| $2P_{3/2}$ | 2 | 1 | ½ | 3/2 | 435 | 408 |

" " " "

all in eV

## 2.3 Darwin term

Norms (Foot Tab. 2.1 and 2.2)

for $n, l, m$ $\quad \langle \psi | \psi \rangle = \langle R(r) Y(\theta,\phi) \,|\, R(r) Y(\theta,\phi) \rangle$

$$= \langle R(r) | R(r) \rangle \langle Y | Y \rangle_{2\pi\pi}$$

$$= \int_0^\infty r^2 R^*_{nl}(r) R \, dr \cdot \int_0 \int_0 Y^*_{lm} Y_{lm} \sin(\theta) \, d\theta \, d\phi$$

$$= 1 \cdot 1$$

E field of point charge $\quad \vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{e}_r$

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\phi} \frac{\partial}{\partial \phi}$$

$$\vec{E} \cdot \vec{\nabla} = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r}$$

$$\Delta E = \langle \psi | \hat{H}_{Darwin} | \psi \rangle$$

$$= \langle R \cdot Y | -\frac{\hbar^2}{4m^2c^2} \frac{Ze^2}{4\pi\varepsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} | R \cdot Y \rangle$$

$$= -\frac{\hbar^2}{4m^2c^2} \frac{Ze^2}{4\pi\varepsilon_0} \langle R | \frac{1}{r^2} \frac{\partial}{\partial r} | R \rangle$$

$$= \quad " \quad \int^\infty dr \frac{r^2}{r^2} R(r) \frac{\partial}{\partial r} R(r)$$

$$= \quad '' \quad \int_0 dr \frac{r}{r^2} R(r) \frac{\partial}{\partial r} R(r)$$

$$= \quad '' \quad \int_0^\infty dr \, \frac{1}{2} \frac{\partial}{\partial r} R^2(r) \quad *$$

$$= \quad '' \quad \left[ \underbrace{\frac{1}{2} |R(\infty)|^2}_{0} - \frac{1}{2} |R^2(0)|^2 \right]$$

$$= \frac{\pi \hbar^2}{2m^2c^2} \frac{ze^2}{4\pi\varepsilon_0} |\psi(0)|^2$$

- $$2 f(x) \frac{\partial f(x)}{\partial x} = \frac{\partial (f^2(x))}{\partial x} \quad *$$
- $$Y^{l=0}_{m=0} = \frac{1}{\sqrt{4\pi}}$$