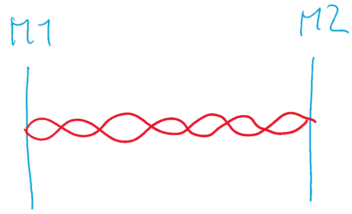
ABCD Matrix Formalism

Dienstag, 23. Mai 2023 12:26

Paraxial ray optics



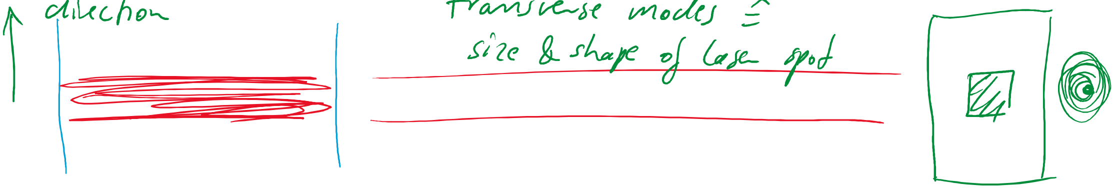
← axial mode ⟶ axis

axial cavity modes

transverse direction

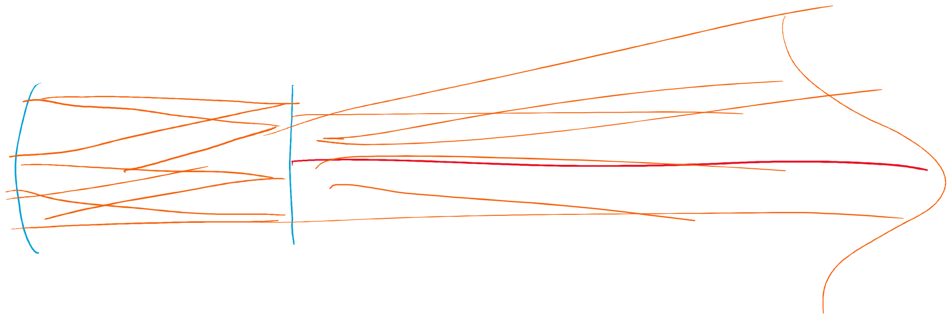
transverse modes ≙ size & shape of laser spot

screen

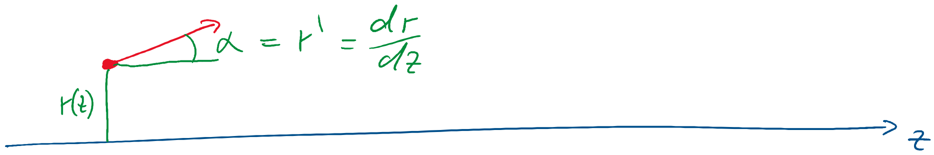


Let's try raytracing

→ no more gain → bad



Raytracing: simplified for rotational symmetry.
can use $x(z)$, $y(z)$ for astigmatic beams



$\alpha = r' = \frac{dr}{dz}$

paraxial approximation : α = small

$\rightarrow \alpha \approx \sin\alpha \approx \tan\alpha = \frac{dr}{dz} = r'$

paraxial approximation : α = small

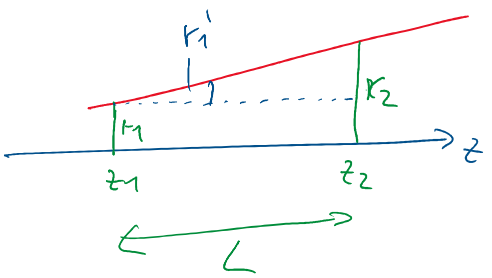
$$\Rightarrow \alpha \approx \sin\alpha \approx \tan\alpha = \frac{dr}{dz} = r'$$

Ray $\rightarrow$ vector $\begin{pmatrix} r \\ \underbrace{r'}_{=\alpha} \end{pmatrix}$ position / slope

<u>examples</u> :

empty space   propagation along z axis for length L

$$z_2 = z_1 + L$$



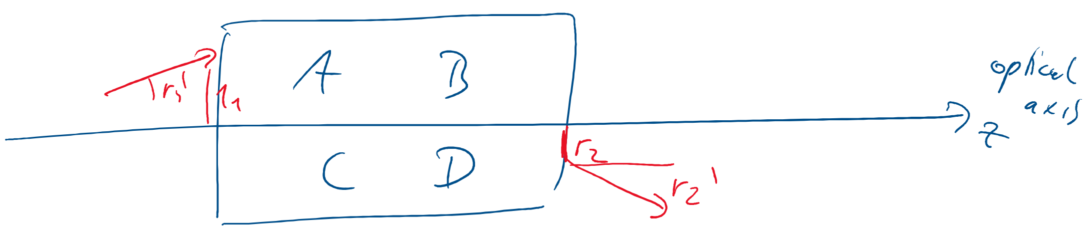
$$\boxed{\begin{aligned} r_2 &= r_1 + L \cdot \overbrace{r_1'}^{=\frac{dr}{dz}} \\ r_2' &= r_1' \end{aligned}}$$

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \boxed{\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix} = \begin{pmatrix} = 1 \cdot r_1 + L \cdot r_1' \\ = 0 \cdot r_1 + 1 \cdot r' \end{pmatrix} \text{o.k.}$$

ray transfer matrix   for empty space

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ matrix}$$

$L \mathrel{\hat=}$ optical element



$$r_2 = A\cdot r_1 + B\cdot r_1'$$
$$r_2' = C\cdot r_1 + D\cdot r_1' \qquad \hat{=} \qquad \vec{r}_2 = M\cdot \vec{r}_1$$

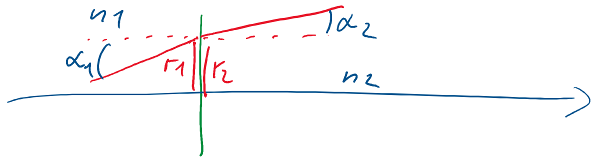
- translation length L

$$M_{Trans} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

- Refraction at optical interface $n_1 \rightarrow n_2$

$$M_{Refr.} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \qquad r_2 = r_1$$

← Snellius: r'

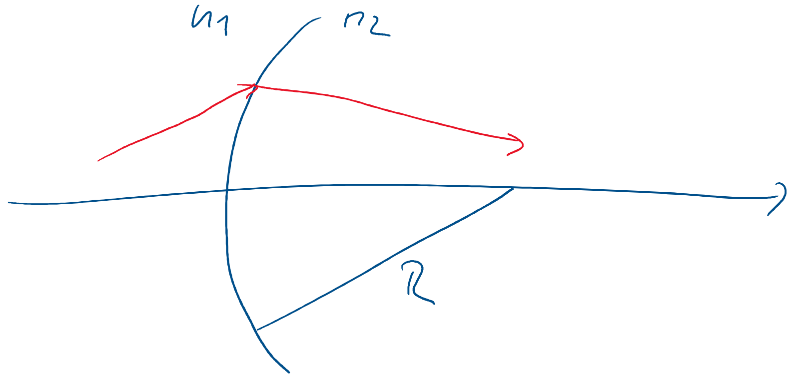


Snell $\quad n_1 \sin\alpha_1 = n_2\, \alpha_2$

paraxial: α = small $\Rightarrow \quad n_1\alpha_1 \approx n_2 \alpha_2$

$$\rightarrow n_1 r_1' = n_2 r_2' \Rightarrow r_2' = r_1' \cdot \frac{n_1}{n_2}$$

- Refraction at curved opt. interface (spherical lens)



$$M_{curved} = \begin{pmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\cdot\rho & \frac{n_1}{n_2} \end{pmatrix}$$

ρ = local curvature of interface

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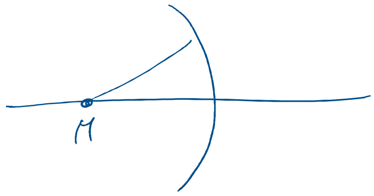
$\rho = \frac{1}{R}$ for spherical interface

$\rho > 0$ convex center is behind surface

$\rho < 0$ concave

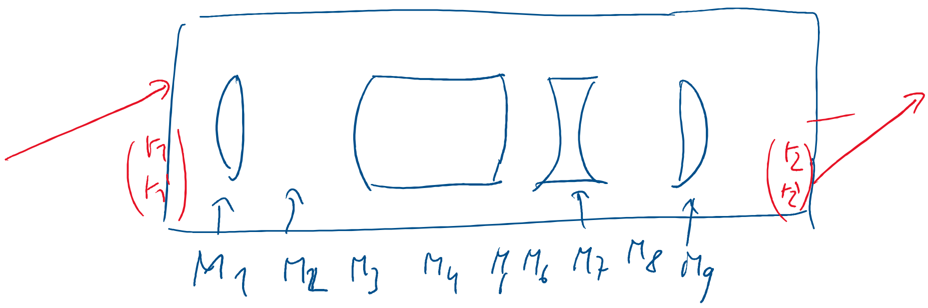


Lens (thin) : $M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

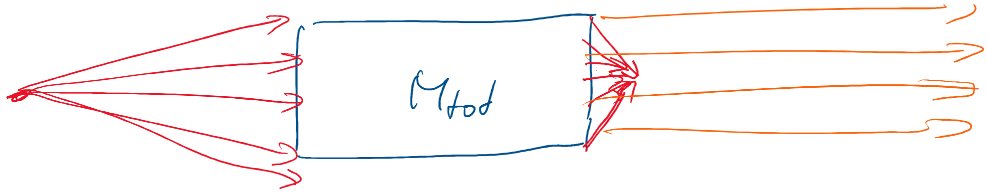
(+ space +)

What is all of this good for?

many optical elements → matrix multiplication yields equivalent single element



$M_{total} = M_9 \cdot M_8 \cdot M_7 \cdot M_6 \cdots\cdots M_2 \cdot M_1$ reverse order



$\vec{r_2} = M_{tot} \cdot \vec{r_1}$

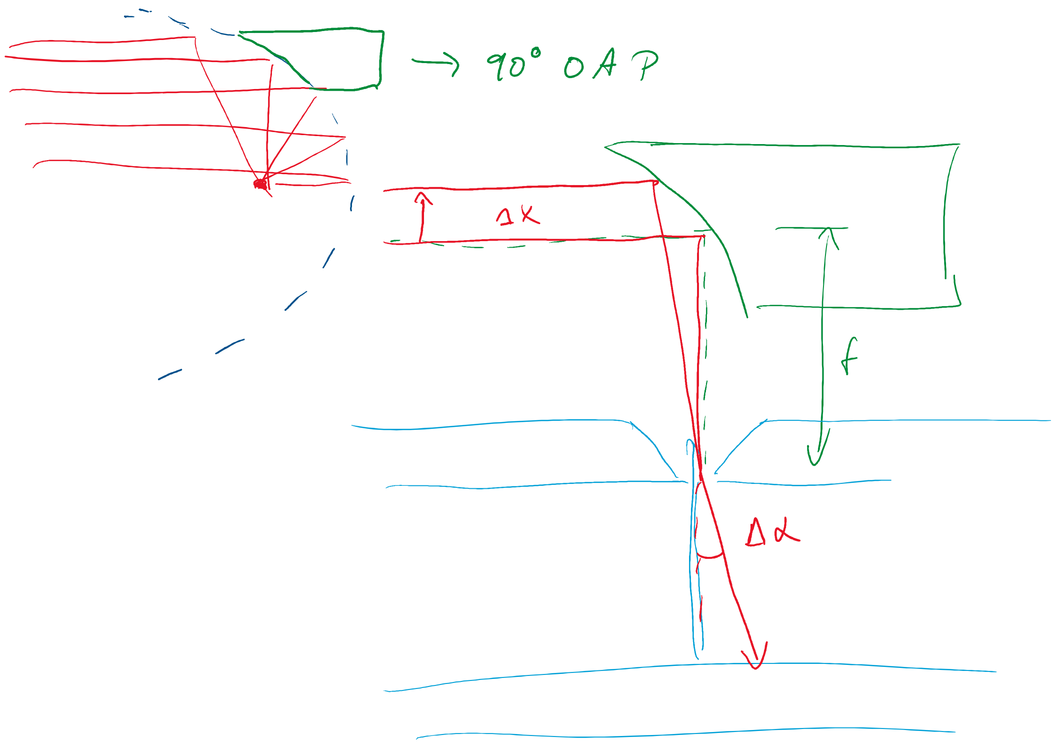
Ray tracing: calculate M_{tot} once

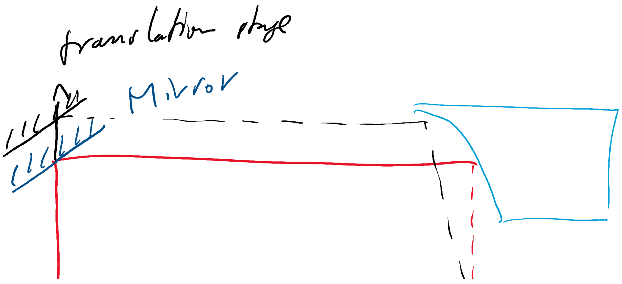
then 4 calculations per ray (many)

$$r_2 = A r_1 + B r_1'$$
$$r_2' = C r_1 + D r_1'$$

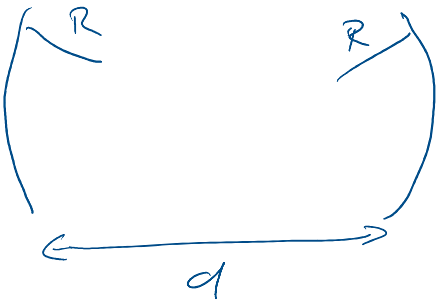
Intermezzo: my cavity

off-axis parabolic mirror





ABCD = good stability analysis of resonators



here: same R
for simplicity

equivalent:

curved mirror $\hat{=}$ lens

$f = R/2$ $\qquad$ $f = R/2$

1 round trip

$$M_{1\,\text{roundtrip}} = M_{\text{lens}} \cdot M_{\text{space}}$$ inverted order

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d \\ -\frac{1}{f} & 1-\frac{d}{f} \end{pmatrix}$$

Eigenvalue problem "eigenray"

$$\vec{r}_{n+1} = M \cdot \vec{r}_n$$

$$M \cdot \vec{r}_1 = \vec{r}_2 = \lambda \cdot \vec{r}_1$$

λ complex number

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$\llcorner$ complex number

$$\left[\overset{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}{M} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \cdot \vec{r} = 0$$ eigen value problem

$$\begin{bmatrix} A-\lambda & B \\ C & D-\lambda \end{bmatrix} \cdot \begin{pmatrix} r \\ r' \end{pmatrix} = 0$$

$$\det \left[M - \lambda I \right] = 0$$

$$\lambda^2 - tr(M)\lambda + \det(M) = 0$$

$$tr(M) = A + D = 2 - \frac{d}{f}$$

$$\det(M) = AD - BC = 1$$

Subst: g = stability parameter of resonator

$$2g = tr(M) = A + D = 2 - \frac{d}{f}$$

$$\Rightarrow \quad \lambda^2 - 2g\lambda + 1 = 0$$

$$\lambda_\pm = g \pm \sqrt{g^2 - 1}$$

Stability: ray goes N times through system

$$\vec{r_n} = \lambda^N \cdot \vec{r_1}$$

stability means: neither r_n nor r_n' are "too large"

$$\Rightarrow \boxed{g^2 \leq 1}$$

(if $g^2 > 1$

λ_+, λ_- are real

$\lambda_+ \cdot \lambda_- = 1$)

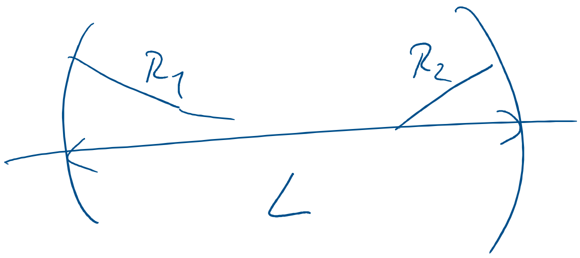
λ_+, λ_- are real

$\lambda_+ \cdot \lambda_- = 1$

$\Rightarrow \lambda_+$ or $\lambda_- > 1$

$\Rightarrow$ divergent

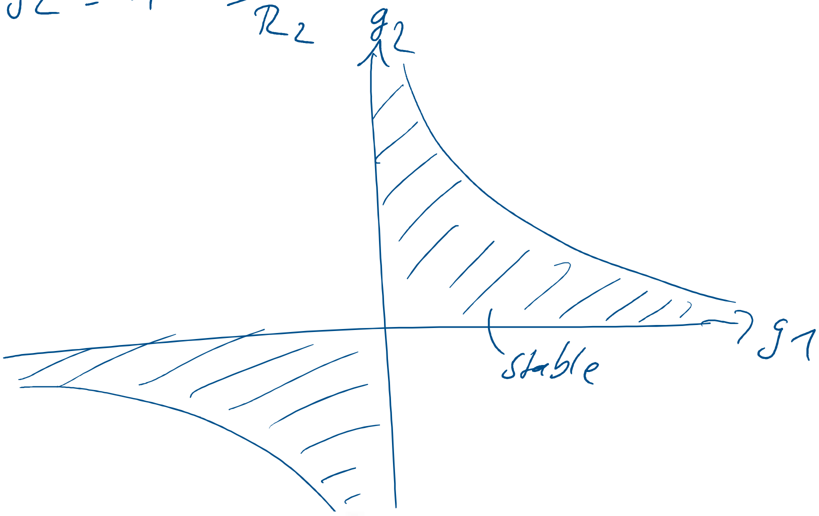
Generalized

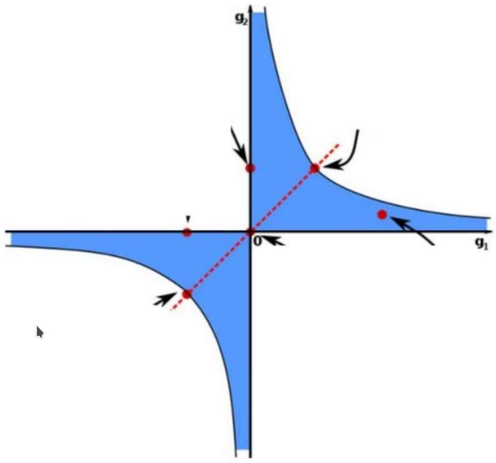


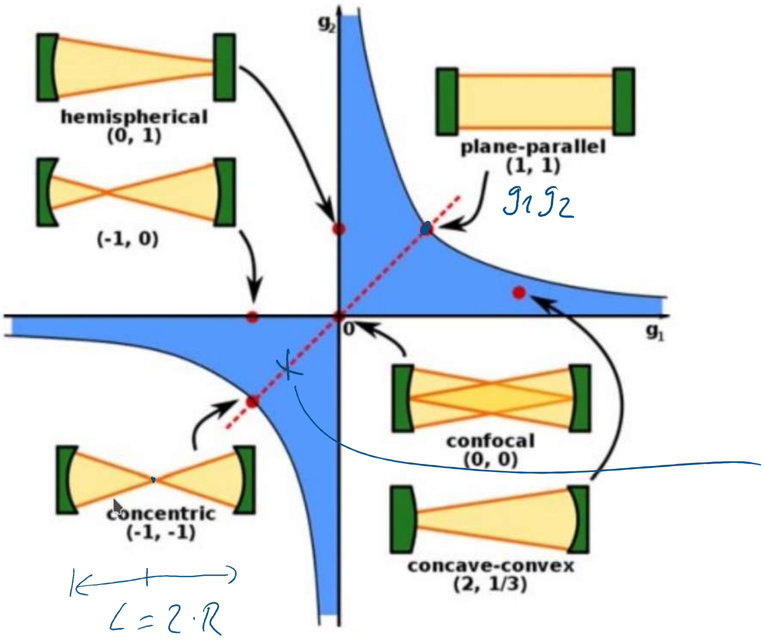
$$g_1 = 1 - \frac{L}{R_1}$$

$$g_2 = 1 - \frac{L}{R_2}$$

$$g_1 \cdot g_2 \leq 1 \rightarrow \text{stable}$$

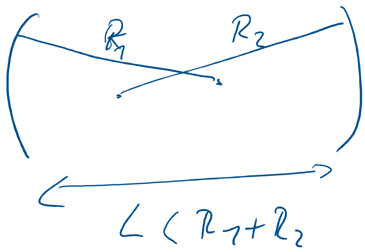


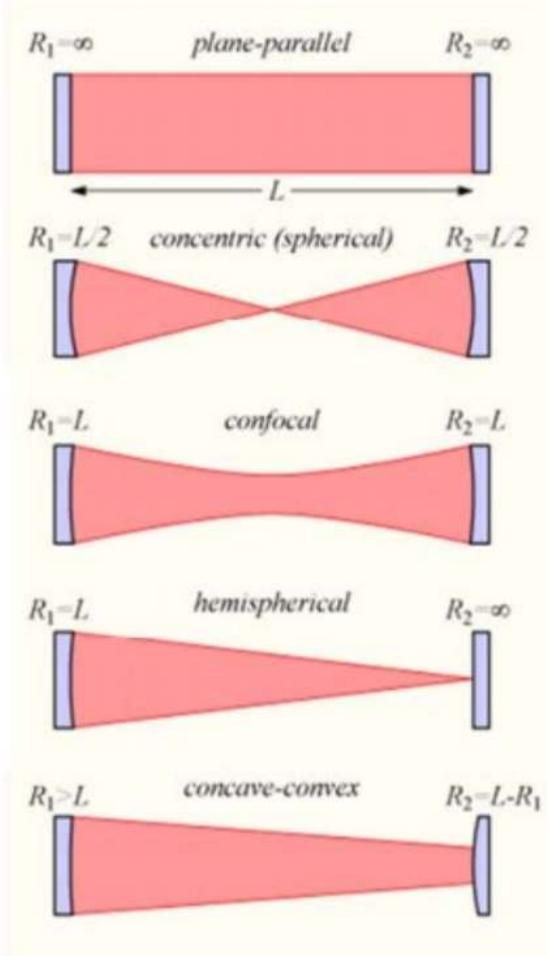




$$g_1 = 1 - \frac{L}{R_1}$$

$$g_2 = 1 - \frac{L}{R_2}$$



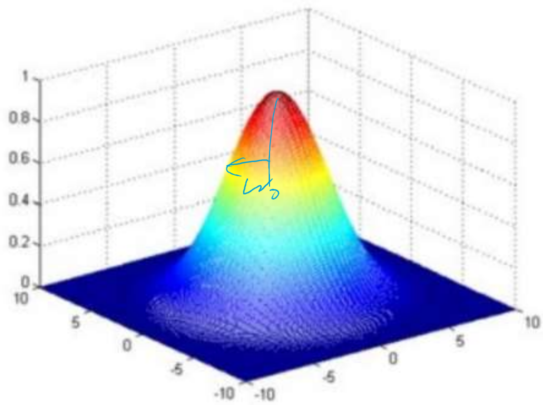


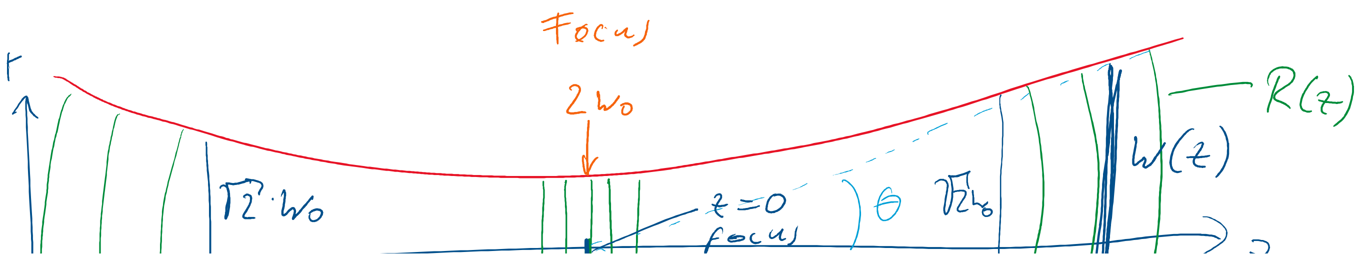
Real beams are not Rays but Gaussians

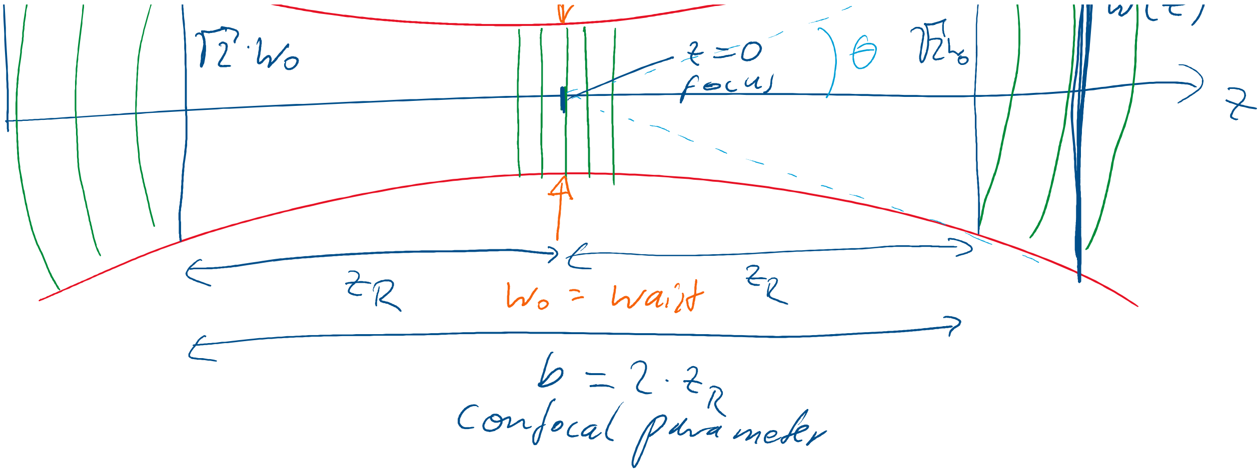
↑ incl. higher order etc.

Fundamental Gaussian bea

2D Gaussian







$$\theta = \frac{\lambda}{\pi w_0} \quad \text{divergence}$$

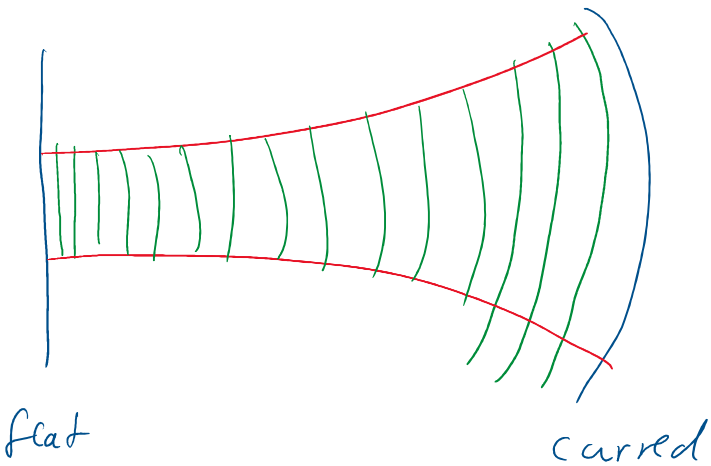
$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$

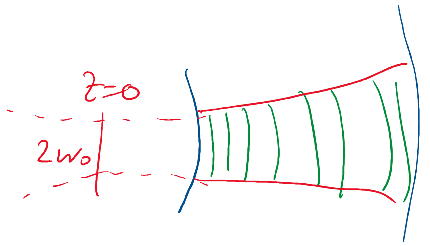
Rayleigh length $z_R = \dfrac{\pi w_0^2}{\lambda}$ ← wavelength

wave front Radius of curvature

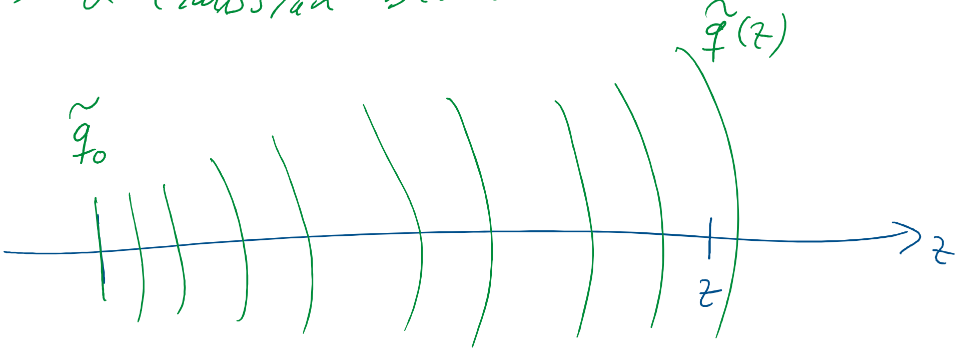
$$R(z) = z \cdot \left[1 + \left(\frac{z_R}{z}\right)^2\right]$$

Stable resonator : $\underline{R_{mirror} = R_{beam}}$





ABCD & Gaussian beams



Complex radius of curvature $\tilde{q}$

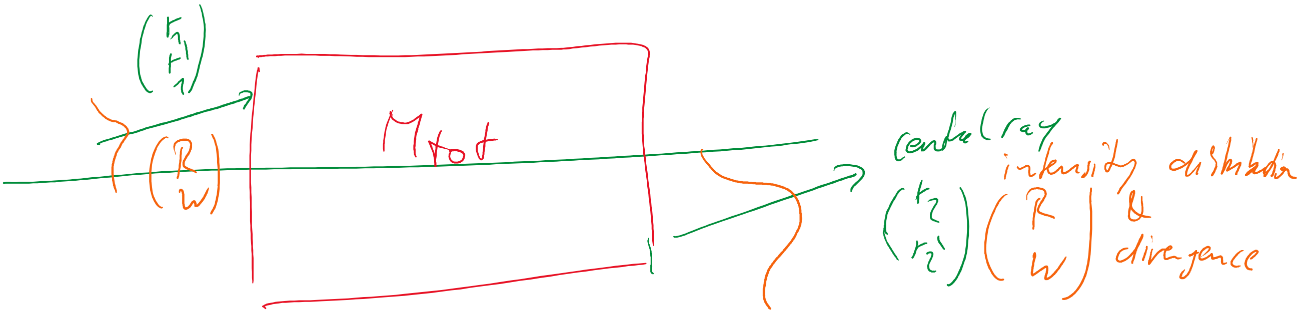
$$\frac{1}{\tilde{q}(z)} = \frac{1}{\underline{R(z)}} - i\,\frac{\lambda}{\pi\,\underline{w(z)}^2}$$

$$\tilde{q}_2/n_2 = \frac{A\cdot \tilde{q}_1/n_1 + B}{C\cdot \tilde{q}_1/n_1 + D}$$

$n_1 \cdot \lambda_1 = n_2 \cdot \lambda_2$

refractive index

These are the same ABCDs as before!



Gouy phase shift : π phase shift

additional phase going through a focus

additional phase going through a focus